

CYCLIC HOMOLOGIES OF CROSSED MODULES OF ALGEBRAS

GURAM DONADZE, NICK INASSARIDZE, EMZAR KHMALADZE, AND MANUEL LADRA

ABSTRACT. The Hochschild and (cotriple) cyclic homologies of crossed modules of (not-necessarily-unital) associative algebras are investigated. Wodzicki's excision theorem is extended for inclusion crossed modules in the category of crossed modules of algebras. The cyclic and cotriple cyclic homologies of crossed modules are compared in terms of long exact homology sequence, generalising the relative cyclic homology exact sequence.

1. INTRODUCTION

The present paper is concerned with the Hochschild and cyclic homology theories of crossed modules of associative algebras, or equivalently, of simplicial associative algebras with the associated Moore complex of length 1.

The study of (co)homological properties of similar objects in the category of groups has been the subject of several papers, for instance, the works of Baues [2] and Ellis [9] investigating the (co)homology of crossed modules of groups as the (co)homology of their classifying spaces; the work of Carrasco, Cegarra and Grandjeán [5] making the observation to the same subject but in the cotriple (co)homology point of view; and [10] giving a connection between the cotriple homology of crossed modules and the homology of their classifying spaces.

Crossed modules of groups were introduced by Whitehead in the late 1940s as algebraic models for path-connected CW-spaces whose homotopy groups are trivial in dimensions > 2 [18]. Since their introduction crossed modules have played an important role in homotopy theory. For illustration we mention various classification problems for low-dimensional homotopy types and derivation of van Kampen theorem generalisations (see the survey of Brown [4]).

Crossed modules of Lie and associative algebras have also been investigated by various authors. Namely, in the works of Dedecker and Lue [6, 16] crossed modules of associative algebras have played a central role in what must be coefficients in low-dimensional non-abelian cohomology. In [3] Baues and Minian have shown that crossed modules of

1991 *Mathematics Subject Classification.* 17B40, 17B56, 18G10, 18G50, 18G60, 19D55.

Key words and phrases. cyclic homology, non-abelian derived functors, simplicial algebra.

The first three authors would like to thank the *University of Santiago de Compostela* for its hospitality during the work on this paper. They were also partially supported by GNSF/ST06/3-004. The authors were partially supported by INTAS grant 06-1000017-8609. The second and the forth authors were supported by MEC grant MTM 2006-15338-C02 (European FEDER support included) and by Xunta de Galicia grant PGIDITI06PXIB371128PR..

associative algebras can be used to represent the Hochschild cohomology. In [14] Kassel and Loday have used crossed modules of Lie algebras to characterize the third Chevalley-Eilenberg cohomology of Lie algebras.

The aim of this paper is to construct and study the cotriple cyclic homology of crossed modules of (non-unital) associative algebras, generalising the classical cyclic homology of associative algebras in zero characteristic case, and to compare it with the cyclic homology of their nerves in terms of long exact homology sequence.

1.1. Organisation. After the introductory Section 1, the paper is organized in four sections. Section 2 is devoted to recalling some necessary definitions and results about crossed modules of associative algebras and the Hochschild and cyclic homologies of simplicial associative algebras. We begin Section 3 by examining the Hochschild and cyclic homologies of aspherical augmented simplicial algebras (Proposition 3.1.1 and Corollary 3.1.2). Then we give the five-term exact sequences relating the Hochschild and cyclic homologies of crossed modules and algebras in low dimensions (Theorem 3.3.1). Finally in this section, we investigate the excision problem for Hochschild (resp. cyclic) homology of inclusion crossed modules of algebras (Theorem 3.4.1). In Section 4 we construct and study the cotriple cyclic homology theory in the category of crossed modules of associative algebras. Then, we calculate the cotriple cyclic homology of inclusion crossed modules as the relative cyclic homology (Proposition 4.2.4). In Section 5 we compare the cyclic and cotriple cyclic homology theories of crossed modules of associative algebras in terms of long exact homology sequence (Theorem 5.0.6).

1.2. Notations and Conventions. We fix \mathbf{k} as a ground field. We make no assumptions on the characteristic of \mathbf{k} , except as stated. All tensor products are over \mathbf{k} . Moreover, $A^{\otimes n} = A \otimes \cdots \otimes A$, n factors. Vectorspaces are considered over \mathbf{k} and their category is denoted by \mathbf{Vect} , while $\mathfrak{C}_{\geq 0}$ is the category of non-negatively graded complexes of vectorspaces. Algebras are (non-unital) associative algebras over \mathbf{k} and their category is denoted by \mathbf{Alg} . The term *free algebra* means a free (non-unital) algebra over some vectorpace. Ideals are always two-sided.

For any functor $T : \underline{\mathcal{C}} \rightarrow \mathbf{Vect}$, where $\underline{\mathcal{C}}$ coincides with the category \mathbf{Alg} or the category of crossed modules of algebras, and for any simplicial object C_* in $\underline{\mathcal{C}}$ denote by $T(C_*)$ the simplicial vectorpace obtained by applying the functor T dimension-wise to C_* .

2. PRELIMINARIES

2.1. Action of algebras and semidirect product. Let A and R be two algebras. By an *action* of A on R we mean an A -bimodule structure on R satisfying the following conditions:

$$a(rr') = (ar)r', \quad (ra)r' = r(ar'), \quad (rr')a = r(r'a)$$

for all $a \in A$, $r, r' \in R$. For example, if R is an ideal of the algebra A , then the multiplication in A yields an action of A on R .

Given an algebra action of A on R , denote by $[A, R]$ the sub-vectorspace of R generated by the elements $[a, r] = ar - ra$ for $r \in R$, $a \in A$. Moreover, one can form *the semidirect product algebra*, $R \rtimes A$, with the underlying vectorspace $R \oplus A$ endowed with the multiplication given by

$$(r, a)(r', a') = (rr' + ar' + ra', aa')$$

for $(r, a), (r', a') \in R \rtimes A$.

2.2. Crossed module and its nerve. Now we recall the basic notions about crossed modules of algebras (cf. [7] and [8]).

A *crossed module* (R, A, ρ) of algebras is an algebra homomorphism $\rho : R \rightarrow A$, together with an action of A on R , such that the following conditions hold:

$$\begin{aligned} \rho(ar) &= a\rho(r), & \rho(ra) &= \rho(r)a, \\ \rho(r)r' &= rr' = r\rho(r') \end{aligned} \quad (\text{Peiffer identity})$$

for all $a \in A$, $r, r' \in R$. We point out that the image of ρ is necessarily an ideal of A , and that $\ker \rho$, contained in the two-sided annihilator of R , is an $A/\rho R$ -bimodule.

The concept of a crossed module of algebras generalizes the concepts of an ideal as well as a bimodule. In fact, a common instance of a crossed module of algebras is that of an algebra A possessing an ideal I ; the inclusion homomorphism $I \hookrightarrow A$ is a crossed module with A acting on I by the multiplication in A , called *the inclusion crossed module* of algebras.

Another common instance is that of an A -bimodule M with trivial multiplication; then the zero homomorphism $0 : M \rightarrow A$, $m \mapsto 0$, is a crossed module.

Any epimorphism of algebras $R \twoheadrightarrow A$ with the kernel in the two-sided annihilator of R is a crossed module, with $a \in A$ acting on $r \in R$ by $ar = \tilde{r}r$ and $ra = r\tilde{r}$, where \tilde{r} is any element in the preimage of a .

A *morphism* $(\mu, \nu) : (R, A, \rho) \rightarrow (R', A', \rho')$ of crossed modules is a commutative square of algebras

$$\begin{array}{ccc} R & \xrightarrow{\mu} & R' \\ \rho \downarrow & & \downarrow \rho' \\ A & \xrightarrow{\nu} & A' \end{array}$$

such that $\mu(ar) = \nu(a)\mu(r)$ and $\mu(ra) = \mu(r)\nu(a)$ for $a \in A$, $r \in R$. Let us denote the category of crossed modules of algebras by $\mathcal{X}\mathbf{Alg}$.

Given a crossed module (R, A, ρ) of algebras, consider the semidirect product algebra, $R \rtimes A$. There are algebra homomorphisms $\mathfrak{s} : R \rtimes A \rightarrow A$, $(r, a) \mapsto a$ and $\mathfrak{t} : R \rtimes A \rightarrow A$, $(r, a) \mapsto \rho(r) + a$ and binary operation $(r', a') \circ (r, a) = (r + r', a)$ for all pairs $(r, a), (r', a') \in R \rtimes A$ such that $\rho(r) + a = a'$. This composition \circ with the *source* map \mathfrak{s} and *target* map \mathfrak{t} constitutes an internal category in the category \mathbf{Alg} and the nerve of its category structure forms the simplicial algebra $\mathfrak{N}_*(R, A, \rho)$, where $\mathfrak{N}_n(R, A, \rho) = R \rtimes (\cdots (R \rtimes A) \cdots)$ with

n semidirect factors of R , and face and degeneracy homomorphisms are defined by

$$\begin{aligned} d_0(r_1, \dots, r_n, a) &= (r_2, \dots, r_n, a), \\ d_i(r_1, \dots, r_n, a) &= (r_1, \dots, r_i + r_{i+1}, \dots, r_n, a), \quad 0 < i < n, \\ d_n(r_1, \dots, r_n, a) &= (r_1, \dots, r_{n-1}, \rho(r_n) + a), \\ s_i(r_1, \dots, r_n, a) &= (r_1, \dots, r_i, 0, r_{i+1}, \dots, r_n, a), \quad 0 \leq i \leq n. \end{aligned}$$

The simplicial algebra $\mathfrak{N}_*(R, A, \rho)$ is called *the nerve of the crossed module* (R, A, ρ) .

2.3. Homologies of simplicial algebras. Let us recall that for a given simplicial algebra A_* its Moore Normalisation is a complex of algebras $\mathcal{N}A_*$, where

$$\mathcal{N}_n A_* = \bigcap_{i=0}^{n-1} \text{Ker } d_i^n \quad \text{and} \quad \partial_n = d_n^n|_{\mathcal{N}_n A_*}.$$

Note that the Moore complex of the nerve of a crossed module of algebras (R, A, ρ) is trivial in dimensions ≥ 2 and is just the original crossed module up to isomorphism with R in dimension 1 and A in dimension 0.

The n -th homotopy of the simplicial algebra is defined as $\pi_n(A_*) = \text{Ker } \partial_n / \text{Im } \partial_{n+1}$. Moreover, if it is given an augmented simplicial algebra (A_*, d_0^0, A) , we calculate the extended homotopy groups as $\pi_0(A_*, d_0^0, A) = \text{Ker } d_0^0 / \text{Im } \partial_1$ and $\pi_{-1}(A_*, d_0^0, A) = A / \text{Im } d_0^0$. We say that the augmented simplicial algebra (A_*, d_0^0, A) is aspherical if $\pi_n(A_*, d_0^0, A) = 0$ for all $n \geq -1$. It is well-known that in any homotopy group π_n , $n \geq 1$, the multiplication induced by that of A_n vanishes.

Given an algebra A , the standard bar, $C^{bar}(A)$, and Hochschild, $C(A)$, complexes have the form

$$C_n^{bar}(A) = C_n(A) := A^{\otimes(n+1)},$$

where the boundary operator of the bar complex is given by

$$b'(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n),$$

while the Hochschild boundary is given by

$$b(a_0 \otimes \dots \otimes a_n) = b'(a_0 \otimes \dots \otimes a_n) + (-1)^n (a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}).$$

Consider the *cyclic*, first quadrant bicomplex:

$$(1) \quad \begin{array}{ccccc} & b \downarrow & & -b' \downarrow & & b \downarrow \\ A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} \\ & b \downarrow & & -b' \downarrow & & b \downarrow \\ & A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} \end{array},$$

where $t : A^{\otimes(n+1)} \rightarrow A^{\otimes(n+1)}$, $n \geq 0$ is the cyclic operator given by $t(a_0, \dots, a_n) = (-1)^n (a_n, a_0, \dots, a_{n-1})$ and $N : A^{\otimes(n+1)} \rightarrow A^{\otimes(n+1)}$ is the operator defined by $N =$

$1 + t + t^2 + \cdots + t^n$. We denote by $CC(A)$ and $CC^{\{2\}}(A)$ the total complexes of the bicomplexes (1) and that of obtained through deleting all columns whose numbers are ≥ 2 in (1), respectively.

Now suppose that we are given a functorial chain complex $\Phi(A)$, as in the case of $C(A)$, $C^{bar}(A)$, $CC^{\{2\}}(A)$ and $CC(A)$ complexes, and set $H_n^\Phi(A) = H_n(\Phi(A))$, $n \geq 0$. Then, extending these homology to simplicial algebras in a usual way, for a given simplicial algebra A_* , denote by $\Phi(A_*)$ the following bicomplex:

$$\begin{array}{ccccccc} & d_2 \downarrow & & -d_2 \downarrow & & d_2 \downarrow & \\ \Phi_1(A_0) & \longleftarrow & \Phi_1(A_1) & \longleftarrow & \Phi_1(A_2) & \longleftarrow & \\ & d_1 \downarrow & & -d_1 \downarrow & & d_1 \downarrow & \\ \Phi_0(A_0) & \longleftarrow & \Phi_0(A_1) & \longleftarrow & \Phi_0(A_2) & \longleftarrow & , \end{array}$$

where horizontal differentials are obtained by taking alternating sums, and by $H_n^\Phi(A_*)$ the n -th homology of its total complex $\text{Tot}(\Phi(A_*))$ (e.g. see [11]).

Given a crossed module of algebras (R, A, ρ) , denote by $\Phi(R, A, \rho)$ the total complex $\text{Tot}(\Phi(\mathfrak{N}_*(R, A, \rho)))$. Then the *Hochschild*, *bar*, *naive Hochschild* and *cyclic* homology of the crossed module (R, A, ρ) are defined by

$$\begin{aligned} HH_n(R, A, \rho) &= H_n(CC^{\{2\}}(R, A, \rho)), & H_n^{bar}(R, A, \rho) &= H_n(C^{bar}(R, A, \rho)), \\ HH_n^{naive}(R, A, \rho) &= H_n(C(R, A, \rho)) \quad \text{and} \quad HC_n(R, A, \rho) &= H_n(CC(R, A, \rho)), \end{aligned}$$

respectively, for $n \geq 0$.

2.4. Linearly split extension. Let (R, A, ρ) , (S, B, σ) and (T, C, θ) be crossed modules of algebras and

$$(2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{\mu} & S & \xrightarrow{\mu'} & T \longrightarrow 0 \\ & & \rho \downarrow & & \downarrow \sigma & & \downarrow \theta \\ 0 & \longrightarrow & A & \xrightarrow{\nu} & B & \xrightarrow{\nu'} & C \longrightarrow 0, \end{array}$$

a sequence of crossed modules. (2) is called an extension of crossed modules if both of its rows are exact sequences of algebras. An extension (2) of crossed modules of algebras is called a linearly split extension if there exists a pair of linear maps $\gamma : T \rightarrow S$ and $\delta : C \rightarrow B$ such that $\mu'\gamma = 1_T$, $\nu'\delta = 1_C$ and $\sigma\gamma = \delta\theta$.

3. HOCHSCHILD AND CYCLIC HOMOLOGY OF CROSSED MODULES

3.1. Homologies of aspherical augmented simplicial algebras. We begin this section with a few results about homology of aspherical augmented simplicial algebras which we shall need in the sequel.

Proposition 3.1.1. *Let (A_*, d_0^0, A) be an aspherical augmented simplicial algebra and $\Phi : \mathbf{Alg} \rightarrow \mathfrak{C}_{\geq 0}$ be a covariant functor, that for each $n \geq 0$, there is a functor $\tilde{\Phi}_n : \mathbf{Vect} \rightarrow \mathbf{Vect}$ such that the diagram*

$$\begin{array}{ccc} \mathbf{Alg} & \xrightarrow{\Phi_n} & \mathbf{Vect} \\ \mathcal{U} \downarrow & \nearrow \tilde{\Phi}_n & \\ \mathbf{Vect} & & \end{array}$$

commutes, where \mathcal{U} is the forgetful functor from the category \mathbf{Alg} to the category \mathbf{Vect} . Then

(i) *the augmented simplicial vectorspace*

$$(\Phi_n(A_*), \Phi_n(d_0^0), \Phi_n(A))$$

is acyclic for any $n \geq 0$;

(ii) *there is a natural isomorphism*

$$H_n^\Phi(A) \cong H_n^\Phi(A_*), \quad n \geq 0.$$

Proof. (i) Straightforward from the fact that an acyclic augmented simplicial vectorspace $(\mathcal{U}(A_*), \mathcal{U}(d_0^0), \mathcal{U}(A))$ poses a linear left (right) contraction.

(ii) Let us consider the bicomplex $\Phi(A_*)$. Using (i), for any fixed q the (horizontal) homology of the bicomplex $\Phi(A_*)$ is $H_p(\Phi_q(A_*)) = 0$, $p > 0$ and $H_0(\Phi_q(A_*)) = \Phi_q(A)$. Now the bicomplex spectral sequence argument completes the proof. \square

Corollary 3.1.2. *Let (A_*, d_0^0, A) be an aspherical augmented simplicial algebra. Then there are natural isomorphisms*

$$\begin{aligned} HH_n(A_*) &\cong HH_n(A), & HH_n^{bar}(A_*) &\cong HH_n^{bar}(A), \\ HH_n^{naive}(A_*) &\cong HH_n^{naive}(A) \quad \text{and} \quad HC_n(A_*) &\cong HC_n(A) \end{aligned}$$

for any $n \geq 0$.

Proof. It is clear that values of the functors C_n , C_n^{bar} , $CC_n^{\{2\}}$ and CC_n , $n \geq 0$, on the algebra A depend only on the vectorspace underlying A . Thanks to Proposition 3.1.1 (ii) the proof is completed. \square

Note that if we are given an inclusion crossed module of algebras $R \hookrightarrow A$, then $\mathfrak{N}_*(R \hookrightarrow A)$ is an aspherical augmented simplicial algebra and by Corollary 3.1.2 any its homology theory mentioned above coincides with the respective homology of the quotient algebra A/R .

3.2. Connes' Periodicity Exact Sequences. The title of this subsection refers to the Connes' exact sequence extended for crossed modules of algebras, connecting their Hochschild and cyclic homologies in terms of long exact periodic sequence. Namely, we have the following.

Proposition 3.2.1. *Let (R, A, ρ) be a crossed module of algebras. There is a natural long exact sequence*

$$\cdots \rightarrow HH_n(R, A, \rho) \xrightarrow{I} HC_n(R, A, \rho) \xrightarrow{S} HC_{n-2}(R, A, \rho) \xrightarrow{B} HH_{n-1}(R, A, \rho) \rightarrow \cdots$$

Proof. There is a short exact sequence of complexes

$$0 \longrightarrow CC^{\{2\}}(R, A, \rho) \longrightarrow CC(R, A, \rho) \longrightarrow CC(R, A, \rho)_{-2} \longrightarrow 0,$$

where $CC_n(R, A, \rho)_{-2} = CC_{n-2}(R, A, \rho)$. This implies the result. \square

3.3. Five-term exact sequences. Now we establish the five-term exact sequences relating the Hochschild and cyclic homologies of crossed modules of algebras and their cokernel algebras.

Theorem 3.3.1. *Let (R, A, ρ) be a crossed module of algebras. There are exact sequences of vectorspaces*

$$\begin{aligned} HH_2(R, A, \rho) &\longrightarrow HH_2(\text{Coker } \rho) \longrightarrow \text{Ker } \rho/[A, \text{Ker } \rho] \longrightarrow HH_1(R, A, \rho) \\ &\longrightarrow HH_1(\text{Coker } \rho) \longrightarrow 0, \\ HC_2(R, A, \rho) &\longrightarrow HC_2(\text{Coker } \rho) \longrightarrow \text{Ker } \rho/[A, \text{Ker } \rho] \longrightarrow HC_1(R, A, \rho) \\ &\longrightarrow HC_1(\text{Coker } \rho) \longrightarrow 0 \end{aligned}$$

and equality

$$HH_0(R, A, \rho) = HC_0(R, A, \rho) = \text{Coker } \rho/[\text{Coker } \rho, \text{Coker } \rho].$$

Proof. We will only prove the exactness of the first sequence. The proof for the second is essentially the same and left to the reader.

Consider the bicomplex $CC^{\{2\}}(\mathfrak{N}_*(R, A, \rho))$. Then there is a first quadrant spectral sequence

$$E_{pq}^1 = H_q(CC_p^{\{2\}}(\mathfrak{N}_*(R, A, \rho))) \Rightarrow HH_{p+q}(R, A, \rho).$$

It is easy to check that we have

$$E_{p0}^1 = CC_p^{\{2\}}(\text{Coker } \rho), \quad p \geq 0 \quad \text{and} \quad E_{01}^1 = \pi_1(\mathfrak{N}_*(R, A, \rho)) = \text{Ker } \rho.$$

Using the Eilenberg-Zilber Theorem and the Künneth Formula we also have

$$\begin{aligned} E_{11}^1 &= \left(\pi_1(\mathfrak{N}_*(R, A, \rho)) \otimes \pi_0(\mathfrak{N}_*(R, A, \rho)) \right) \oplus \left(\pi_0(\mathfrak{N}_*(R, A, \rho)) \otimes \pi_1(\mathfrak{N}_*(R, A, \rho)) \right) \\ &\oplus \pi_1(\mathfrak{N}_*(R, A, \rho)) = \left(\text{Ker } \rho \otimes \text{Coker } \rho \right) \oplus \left(\text{Coker } \rho \otimes \text{Ker } \rho \right) \oplus \text{Ker } \rho. \end{aligned}$$

Continuing calculations we deduce that

$$E_{00}^\infty = E_{00}^2 = HH_0(\text{Coker } \rho) \quad \text{and} \quad E_{10}^\infty = E_{10}^2 = HH_1(\text{Coker } \rho).$$

Moreover,

$$E_{01}^2 = \text{Coker}\{E_{11}^1 \rightarrow E_{01}^1\} = \text{Ker } \rho/[\text{Ker } \rho, \text{Coker } \rho] = \text{Ker } \rho/[A, \text{Ker } \rho].$$

Therefore, we have a differential $d^2 : HH_2(\text{Coker } \rho) \rightarrow \text{Ker } \rho/[A, \text{Ker } \rho]$ of the spectral sequence, which determines the base term E_{20}^∞ and the fiber term E_{01}^∞ from the following exact sequence:

$$(3) \quad 0 \rightarrow E_{20}^\infty \rightarrow HH_2(\text{Coker } \rho) \rightarrow \text{Ker } \rho/[A, \text{Ker } \rho] \rightarrow E_{01}^\infty \rightarrow 0.$$

Clearly, we have the short exact sequence

$$(4) \quad 0 \rightarrow E_{01}^\infty \rightarrow HH_1(R, A, \rho) \rightarrow E_{10}^\infty \rightarrow 0$$

and the epimorphism

$$(5) \quad HH_2(R, A, \rho) \twoheadrightarrow E_{20}^\infty.$$

Now (3), (4) and (5) imply the required result. \square

This statement shows that the Hochschild (resp. cyclic) homology of a crossed module (R, A, ρ) differs, in general, from the Hochschild (resp. cyclic) homology of the cokernel algebra $\text{Coker } \rho$.

3.4. Excision property. In this subsection we discuss some aspects of the excision property for Hochschild (resp. cyclic) homology of crossed modules. Namely, we give sufficient conditions for inclusion crossed modules of algebras when they satisfy the excision property. The concept of investigation of the problem in general setting according to Wodzicki is substantially different and will be treated in a separate paper.

The excision problem for Hochschild (resp. cyclic) homology in the category of crossed module of algebras is formulated as follows: let

$$(6) \quad 0 \longrightarrow (R, A, \rho) \xrightarrow{(\mu, \nu)} (S, B, \sigma) \xrightarrow{(\eta, \theta)} (T, C, \tau) \longrightarrow 0$$

be a linearly split extension of crossed modules of algebras. The crossed module (R, A, ρ) is excisive (or satisfies excision) for Hochschild (resp. cyclic) homology in the category of crossed modules if the induced natural long homology sequence

$$\cdots \rightarrow HH_n(R, A, \rho) \rightarrow HH_n(S, B, \sigma) \rightarrow HH_n(T, C, \tau) \rightarrow HH_{n-1}(R, A, \rho) \rightarrow \cdots$$

(resp.

$$\cdots \rightarrow HC_n(R, A, \rho) \rightarrow HC_n(S, B, \sigma) \rightarrow HC_n(T, C, \tau) \rightarrow HC_{n-1}(R, A, \rho) \rightarrow \cdots)$$

is exact for any linearly split extension (6) of the crossed module (R, A, ρ) .

Note that according to the Connes' Periodicity Exact Sequence, Proposition 3.2.1, the excision properties for Hochschild and cyclic homologies in the category of crossed modules of algebras are equivalent.

The aim of this subsection is to prove the following.

Theorem 3.4.1. *Let (I, A, inc) be an inclusion crossed module of algebras such that $H_n^{\text{bar}}(I, A, \text{inc}) = 0$, $n \geq 0$. Then (I, A, inc) is excisive for Hochschild homology.*

Proof. Consider any linearly split extension (6) of crossed modules of algebras. Hence we have the short exact sequence

$$(7) \quad 0 \rightarrow \pi_0(\mathfrak{N}_*(I, A, inc)) \xrightarrow{\pi_0(\mathfrak{N}_*(\mu, \nu))} \pi_0(\mathfrak{N}_*(S, B, \sigma)) \\ \xrightarrow{\pi_0(\mathfrak{N}_*(\eta, \theta))} \pi_0(\mathfrak{N}_*(T, C, \tau)) \rightarrow 0.$$

and the isomorphism

$$(8) \quad \pi_1(\mathfrak{N}_*(S, B, \sigma)) \xrightarrow[\cong]{\pi_1(\mathfrak{N}_*(\eta, \theta))} \pi_1(\mathfrak{N}_*(T, C, \tau)).$$

It is also easy to see that we have the following commutative diagram with exact rows of complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(I, A, inc) & \longrightarrow & CC^{\{2\}}(I, A, inc) & \longrightarrow & C^{bar}(I, A, inc)_{-1} \longrightarrow 0 \\ & & \downarrow C(\mu, \nu) & & \downarrow CC^{\{2\}}(\mu, \nu) & & \downarrow C^{bar}(\mu, \nu) \\ 0 & \longrightarrow & \text{Ker}(C(\eta, \theta)) & \longrightarrow & \text{Ker}(CC^{\{2\}}(\eta, \theta)) & \longrightarrow & \text{Ker}(C^{bar}(\eta, \theta))_{-1} \longrightarrow 0, \end{array}$$

where $C_n^{bar}(I, A, inc)_{-1} = C_{n-1}^{bar}(I, A, inc)$ and $\text{Ker}_n\{C^{bar}(\eta, \theta)\}_{-1} = \text{Ker}_{n-1}\{C^{bar}(\eta, \theta)\}$. Clearly, the induced commutative diagram of long exact homology sequences and the five lemma implies that, if $C(\mu, \nu)$ and $C^{bar}(\mu, \nu)$ are quasi-isomorphisms, then so is $CC^{\{2\}}(\mu, \nu)$.

We shall only prove that $C(\mu, \nu)$ is quasi-isomorphic. The proof that $C^{bar}(\mu, \nu)$ is quasi-isomorphic as well can be accomplished essentially in the same way and will be omitted.

We need the following.

Lemma 3.4.2. *Let (S, B, σ) be any crossed module of algebras, then there is an isomorphism*

$$H_q(C_p(\mathfrak{N}_*(S, B, \sigma))) \cong \begin{cases} \bigoplus_{i_0 + \dots + i_p = q} (\pi_{i_0}(\mathfrak{N}_*(S, B, \sigma)) \otimes \dots \otimes \pi_{i_p}(\mathfrak{N}_*(S, B, \sigma))) & \text{for } p \geq 0, 0 \leq q \leq p+1 \\ 0 & \text{for } 0 \leq p < q-1 \end{cases}$$

with $i_0, \dots, i_p = 0$ or 1 . Moreover, through this isomorphism the Hochschild differential behaves as follows:

$$x_0 \otimes x_1 \otimes \dots \otimes x_p \mapsto \sum_{j=0}^{p-1} (-1)^j (x_0 \otimes \dots \otimes x_j x_{j+1} \otimes \dots \otimes x_p) + (-1)^p (x_p x_0 \otimes \dots \otimes x_{p-1}),$$

where $x_j \in \pi_{i_j}(\mathfrak{N}_*(S, B, \sigma))$, $0 \leq j \leq p$, and the multiplication $x_j x_{j'}$ is meant as the multiplication in $\pi_0(\mathfrak{N}_*(S, B, \sigma))$ or the bimodule structure of $\pi_0(\mathfrak{N}_*(S, B, \sigma))$ on $\pi_1(\mathfrak{N}_*(S, B, \sigma))$.

Proof. The proof, requiring to use again the Eilenberg-Zilber Theorem and the Künneth Formula, is routine and will be omitted. \square

Returning to the main proof, consider the bicomplexes $C(\mathfrak{N}_*(I, A, inc))$ and $\mathbb{M} \equiv \text{Ker} \left(C(\mathfrak{N}_*(S, B, \sigma)) \xrightarrow{C(\mathfrak{N}_*(\eta, \theta))} C(\mathfrak{N}_*(T, C, \tau)) \right)$. Then there are first quadrant spectral sequences

$$E_{pq}^1 = H_q(C_p(\mathfrak{N}_*(I, A, inc))) \Rightarrow HH_{p+q}^{naive}(I, A, inc)$$

and

$$\overline{E}_{pq}^1 = H_q \left(\text{Ker} \left(C_p(\mathfrak{N}_*(S, B, \sigma)) \xrightarrow{C_p(\mathfrak{N}_*(\eta, \theta))} C_p(\mathfrak{N}_*(T, C, \tau)) \right) \right) \Rightarrow H_{p+q}(\text{Tot}(\mathbb{M})).$$

Moreover, we have the linearly split exact sequence of simplicial vectorspaces

$$0 \rightarrow \mathbb{M}_{*p} \rightarrow C_p(\mathfrak{N}_*(S, B, \sigma)) \rightarrow C_p(\mathfrak{N}_*(T, C, \tau)) \rightarrow 0,$$

implying the short exact homology sequence

$$0 \rightarrow \overline{E}_{pq}^1 \rightarrow H_q(C_p(\mathfrak{N}_*(S, B, \sigma))) \rightarrow H_q(C_p(\mathfrak{N}_*(T, C, \tau))) \rightarrow 0.$$

Hence, from Lemma 3.4.2 we have an isomorphism

$$(9) \quad \overline{E}_{pq}^1 \cong \begin{cases} \bigoplus_{i_0 + \dots + i_p = q} \text{Ker} \{ \pi_{i_0}(\mathfrak{N}_*(\eta, \theta)) \otimes \dots \otimes \pi_{i_p}(\mathfrak{N}_*(\eta, \theta)) \} & \text{for } p \geq 0, 0 \leq q \leq p+1 \\ 0 & \text{for } 0 \leq p < q-1 \end{cases}$$

with $i_0, \dots, i_p = 0$ or 1 .

Clearly, there is a natural morphism of bicomplexes $C(\mathfrak{N}_*(\mu, \nu)) : C(\mathfrak{N}_*(I, A, inc)) \rightarrow \mathbb{M}$, inducing the morphism of spectral sequences $f^1 : E^1 \rightarrow \overline{E}^1$. To finish the proof it suffices to show that $f_{pq}^2 : E_{pq}^2 \rightarrow \overline{E}_{pq}^2$ is an isomorphism for any $p, q \geq 0$.

We prove the remaining part of the assertion in two cases.

Case 1. *The homomorphism $f_{pq}^2 : E_{pq}^2 \rightarrow \overline{E}_{pq}^2$ is an isomorphism for any $p \geq 0$ and $q = 0$.*

In effect, by Lemma 3.4.2, $E_{p0}^2 = HH_p^{naive}(\pi_0(\mathfrak{N}_*(I, A, inc)))$, while by (9) we have

$$\overline{E}_{p0}^2 \cong H_p \left(\text{Ker} \{ C(\pi_0(\mathfrak{N}_*(S, B, \sigma))) \xrightarrow{C(\pi_0(\mathfrak{N}_*(\eta, \theta)))} C(\pi_0(\mathfrak{N}_*(T, C, \tau))) \} \right).$$

By the assumption on the crossed module (I, A, inc) and Corollary 3.1.2 the algebra $\pi_0(\mathfrak{N}_*(I, A, inc))$ is H -unital, then the result follows from Wodzicki's theorem [19] applied to the short exact sequence of algebras (7).

Case 2. *The homomorphism $f_{pq}^2 : E_{pq}^2 \rightarrow \overline{E}_{pq}^2$ is an isomorphism for any $p \geq 0$ and $q > 0$.*

Using Lemma 3.4.2 again, we have $E_{pq}^1 = 0$ and consequently $E_{pq}^2 = 0$ for any $p \geq 0$ and $q > 0$. Thus, we are left to show that $\overline{E}_{pq}^2 = 0$, $p \geq 0$ and $q > 0$. In effect, by (7) and (8) we have the short exact sequence of algebras

$$(10) \quad 0 \longrightarrow \pi_0(\mathfrak{N}_*(I, A, inc)) \longrightarrow \pi_1(\mathfrak{N}_*(S, B, \sigma)) \rtimes \pi_0(\mathfrak{N}_*(S, B, \sigma)) \\ \longrightarrow \pi_1(\mathfrak{N}_*(T, C, \tau)) \rtimes \pi_0(\mathfrak{N}_*(T, C, \tau)) \longrightarrow 0.$$

Hence, the fact that $\pi_0(\mathfrak{N}_*(I, A, inc))$ is H -unital, by Wodzicki's theorem [19], implies that we have the quasi-isomorphism

$$(11) \quad C(\pi_0(\mathfrak{N}_*(I, A, inc))) \rightarrow \text{Ker} \{C(\pi_1(\mathfrak{N}_*(S, B, \sigma)) \rtimes \pi_0(\mathfrak{N}_*(S, B, \sigma))) \rightarrow C(\pi_1(\mathfrak{N}_*(T, C, \tau)) \rtimes \pi_0(\mathfrak{N}_*(T, C, \tau)))\}.$$

Moreover, we have an isomorphisms of vectorspaces

$$(12) \quad \begin{aligned} C_p(\pi_1(\mathfrak{N}_*(S, B, \sigma)) \rtimes \pi_0(\mathfrak{N}_*(S, B, \sigma))) &\cong (\pi_0(\mathfrak{N}_*(S, B, \sigma)))^{\otimes p+1} \\ &\oplus \bigoplus_{i_0+\dots+i_p=1}^{p+1} (\pi_{i_0}(\mathfrak{N}_*(S, B, \sigma)) \otimes \dots \otimes \pi_{i_p}(\mathfrak{N}_*(S, B, \sigma))) \end{aligned}$$

and

$$(13) \quad \begin{aligned} C_p(\pi_1(\mathfrak{N}_*(T, C, \tau)) \rtimes \pi_0(\mathfrak{N}_*(T, C, \tau))) &\cong (\pi_0(\mathfrak{N}_*(T, C, \tau)))^{\otimes p+1} \\ &\oplus \bigoplus_{i_0+\dots+i_p=1}^{p+1} (\pi_{i_0}(\mathfrak{N}_*(T, C, \tau)) \otimes \dots \otimes \pi_{i_p}(\mathfrak{N}_*(T, C, \tau))) \end{aligned}$$

with $i_0, \dots, i_p = 0$ or 1 .

Let us define \mathfrak{D}_p^q and $\overline{\mathfrak{D}}_p^q$ for any $p \geq 0$ and $q > 0$ by the formulas

$$\mathfrak{D}_p^q = \begin{cases} \bigoplus_{i_0+\dots+i_p=q} (\pi_{i_0}(\mathfrak{N}_*(S, B, \sigma)) \otimes \dots \otimes \pi_{i_p}(\mathfrak{N}_*(S, B, \sigma))) & \text{for } q \leq p+1 \\ 0 & \text{for } p+1 < q \end{cases}$$

and

$$\overline{\mathfrak{D}}_p^q = \begin{cases} \bigoplus_{i_0+\dots+i_p=q} (\pi_{i_0}(\mathfrak{N}_*(T, C, \tau)) \otimes \dots \otimes \pi_{i_p}(\mathfrak{N}_*(T, C, \tau))) & \text{for } q \leq p+1 \\ 0 & \text{for } p+1 < q \end{cases}$$

with $i_0, \dots, i_p = 0$ or 1 .

It is easy to show that \mathfrak{D}^q is a sub-complex of $C(\pi_1(\mathfrak{N}_*(S, B, \sigma)) \rtimes \pi_0(\mathfrak{N}_*(S, B, \sigma)))$ through the isomorphism (12) and along the Hochschild differential, which is left to the reader (hint: the induced multiplication in $\pi_1(\mathfrak{N}_*(S, B, \sigma))$ vanishes). Similarly, $\overline{\mathfrak{D}}^q$ is a sub-complex of $C(\pi_1(\mathfrak{N}_*(T, C, \tau)) \rtimes \pi_0(\mathfrak{N}_*(T, C, \tau)))$. Moreover, by (12) and (13), there are decompositions in direct summands of complexes

$$C(\pi_1(\mathfrak{N}_*(S, B, \sigma)) \rtimes \pi_0(\mathfrak{N}_*(S, B, \sigma))) \cong C(\pi_0(\mathfrak{N}_*(S, B, \sigma))) \oplus \bigoplus_{q \geq 1} \mathfrak{D}^q$$

and

$$C(\pi_1(\mathfrak{N}_*(T, C, \tau)) \rtimes \pi_0(\mathfrak{N}_*(T, C, \tau))) \cong C(\pi_0(\mathfrak{N}_*(T, C, \tau))) \oplus \bigoplus_{q \geq 1} \overline{\mathfrak{D}}^q.$$

Hence

$$\begin{aligned} &\text{Ker} \{C(\pi_1(\mathfrak{N}_*(S, B, \sigma)) \rtimes \pi_0(\mathfrak{N}_*(S, B, \sigma))) \rightarrow C(\pi_1(\mathfrak{N}_*(T, C, \tau)) \rtimes \pi_0(\mathfrak{N}_*(T, C, \tau)))\} \\ &\cong \text{Ker} \{C(\pi_0(\mathfrak{N}_*(S, B, \sigma))) \xrightarrow{\pi_0(\mathfrak{N}_*(\eta, \theta))} C(\pi_0(\mathfrak{N}_*(T, C, \tau)))\} \oplus \bigoplus_{q \geq 1} \text{Ker} \{\mathfrak{D}^q \rightarrow \overline{\mathfrak{D}}^q\}. \end{aligned}$$

Now the quasi-isomorphism (11) implies that the complex $\text{Ker}(\mathfrak{D}^q \rightarrow \overline{\mathfrak{D}}^q)$ is acyclic for any $q > 0$, which according to (9) means that $\overline{E}_{pq}^2 = 0$ for any $p \geq 0$ and $q > 0$. \square

4. COTRIPLE CYCLIC HOMOLOGY OF CROSSED MODULES

In this section we assume that \mathbf{k} is a field of characteristic zero.

4.1. Adjunction. We begin by constructing an adjoint pair of functors $\mathbf{Alg} \xrightleftharpoons[U]{F} \mathcal{X}\mathbf{Alg}$.

Assume that the functor $U : \mathcal{X}\mathbf{Alg} \rightarrow \mathbf{Alg}$ assigns to any crossed module (M, R, μ) the direct product $M \times R$. Now, define the functor $F : \mathbf{Alg} \rightarrow \mathcal{X}\mathbf{Alg}$ as follows: for any algebra A , let $F(A)$ denote the inclusion crossed module of algebras $(\overline{A}, A * A, inc)$, where $A * A$ is the coproduct of the algebra A with itself, with inclusions $u_1, u_2 : A \rightarrow A * A$, and \overline{A} is the kernel of the retraction $p_2 : A * A \rightarrow A$ determined by the conditions $p_2 u_2 = 1_A$, $p_2 u_1 = 0$.

Proposition 4.1.1. *The functor F is left adjoint to the functor U .*

Proof. We state that, given an algebra A , the homomorphism

$$(u_1, u_2) : A \rightarrow \overline{A} \times (A * A) = UF(A)$$

is a universal arrow from A to the functor U . Indeed, let (S, B, σ) be a crossed module and $f_S : A \rightarrow S$, $f_B : A \rightarrow B$ defining homomorphisms of any homomorphism $(f_S, f_B) : A \rightarrow S \times B = U(S, B, \sigma)$. Then there is a commutative diagram with split short exact sequences of algebras

$$\begin{array}{ccccc} \overline{A} & \xrightarrow{inc} & A * A & \xrightarrow{p_2} & A \\ \alpha \downarrow & & \gamma \downarrow & & \downarrow f_B \\ S & \xrightarrow{i} & S \rtimes B & \xrightarrow{pr} & B \end{array} \quad ,$$

$\begin{array}{c} \xrightarrow{u_2} \\ \xrightarrow{p_2} \\ \xrightarrow{j} \end{array}$

where γ is defined by $\gamma u_1 = i f_S$ and $\gamma u_2 = j f_B$, and α is the restriction of γ . Let $\beta : A * A \rightarrow B$ be the unique homomorphism satisfying $\beta u_1 = \sigma f_S$ and $\beta u_2 = f_B$. Easy calculations show that $(\alpha, \beta) : (\overline{A}, A * A, inc) \rightarrow (S, B, \sigma)$ is a morphism of crossed modules of algebras, clearly the unique one such that $(\alpha \times \beta)(u_1, u_2) = (f_S, f_B)$. \square

We denote by $W : \mathbf{Alg} \rightarrow \mathbf{Vect}$ the usual forgetful functor and by $T : \mathbf{Vect} \rightarrow \mathbf{Alg}$ its left adjoint functor, carrying any vector space V to the free algebra on it. Composing these two adjunctions,

$$\mathbf{Vect} \xrightleftharpoons[W]{T} \mathbf{Alg} \xrightleftharpoons[U]{F} \mathcal{X}\mathbf{Alg} \quad ,$$

we deduce the following.

Proposition 4.1.2. *The functor $\mathbb{F} = F \circ T : \mathbf{Vect} \rightarrow \mathcal{X}\mathbf{Alg}$, $V \mapsto (\overline{T(V)}, T(V) * T(V), inc)$, is left adjoint to the functor $\mathbb{U} = W \circ U : \mathcal{X}\mathbf{Alg} \rightarrow \mathbf{Vect}$, $(R, A, \rho) \mapsto R \times A$.*

4.2. Construction and elementary properties. It is known due to [7, 13] that the cyclic homology of algebras is described as the non-abelian derived functors of the additive abelianisation functor $Ab^{add} : \mathbf{Alg} \rightarrow \mathbf{Vect}$, $Ab^{add}(A) = A/[A, A]$. To generalize the cyclic homology theory to the category $\mathcal{X}\mathbf{Alg}$ in terms of non-abelian derived functors we need to extend the additive abelianisation functor to this category. By reason of that we look the functor Ab^{add} as a factorisation through the category of Lie algebras \mathbf{Lie} . Explicitly, there is an equality $Ab^{add} = Ab \circ \mathfrak{L}$, where $\mathfrak{L} : \mathbf{Alg} \rightarrow \mathbf{Lie}$ is the classical Liasation functor and $Ab : \mathbf{Lie} \rightarrow \mathbf{Vect}$ is the abelianisation functor of Lie algebras.

Let us construct the natural extension of the functors \mathfrak{L} and Ab to the category $\mathcal{X}\mathbf{Alg}$. First recall from [14] that a crossed module of Lie algebras (M, G, μ) is a Lie homomorphism $\mu : M \rightarrow G$ together with a bilinear map $G \times M \rightarrow M$, $(g, m) \mapsto {}^g m$ satisfying

$$[g, g']m = {}^g(g'm) - g'({}^g m), \quad {}^g[m, m'] = [{}^g m, m'] + [m, {}^g m'],$$

such that the following conditions hold:

$$(i) \mu({}^g m) = [g, \mu(m)], \quad (ii) \mu({}^{(m)} m') = [m, m'] \quad \text{for all } m, m' \in M, g \in G.$$

Denote by $\mathcal{X}\mathbf{Lie}$ the category of crossed modules of Lie algebras. Then there is a naturally defined functor $\mathcal{X}\mathfrak{L} : \mathcal{X}\mathbf{Alg} \rightarrow \mathcal{X}\mathbf{Lie}$ carrying a crossed module of algebras $\rho : R \rightarrow A$ to the crossed module of Lie algebras $\mathfrak{L}(\mu) : \mathfrak{L}(R) \rightarrow \mathfrak{L}(A)$ with ${}^a r = ar - ra$. Moreover, it is known that an abelian group object in the category $\mathcal{X}\mathbf{Lie}$ is just a linear map of vectorspaces and their category is denoted by $\mathcal{X}\mathbf{Vect}$. Now the abelianisation of crossed modules of Lie algebras $\mathcal{X}Ab : \mathcal{X}\mathbf{Lie} \rightarrow \mathcal{X}\mathbf{Vect}$ is left adjoint to the natural embedding functor $\mathcal{X}\mathbf{Vect} \subset \mathcal{X}\mathbf{Lie}$, which is explicitly given by $\mathcal{X}Ab(M, G, \mu) = (M/[G, M], Ab(G), \tilde{\mu})$ where $[G, M]$ denotes the subvectorspaces of the Lie algebra M generated by the elements ${}^g m$ for $g \in G$, $m \in M$, and $\tilde{\mu}$ is the Lie homomorphism induced by μ .

From the aforementioned discussion we arrive to the definition of the additive abelianisation functor of crossed modules of algebras $\mathcal{X}Ab^{add} : \mathcal{X}\mathbf{Alg} \rightarrow \mathcal{X}\mathbf{Vect}$ as $\mathcal{X}Ab^{add} = \mathcal{X}Ab \circ \mathcal{X}\mathfrak{L}$.

Now we are ready to construct the *cotriple cyclic homology of crossed modules of algebras*. We assume the reader is familiar with cotriples and projective classes. See, for example,

[1] and [12, Chapter 2] for the background. The adjoint pair of functors $\mathbf{Vect} \xrightleftharpoons[\mathbb{U}]{\mathbb{F}} \mathcal{X}\mathbf{Alg}$,

constructed in the previous subsection, induces a cotriple $\mathcal{F} = (\mathcal{F}, \delta, \tau)$ in $\mathcal{X}\mathbf{Alg}$ by the obvious way: $\mathcal{F} = \mathbb{F}\mathbb{U} : \mathcal{X}\mathbf{Alg} \rightarrow \mathcal{X}\mathbf{Alg}$, $\tau : \mathcal{F} \rightarrow 1_{\mathcal{X}\mathbf{Alg}}$ is the counit and $\delta = \mathbb{F}u\mathbb{U} : \mathcal{F} \rightarrow \mathcal{F}^2$, where $u : 1_{\mathbf{Vect}} \rightarrow \mathbb{U}\mathbb{F}$ is the unit of the adjunction. Let \mathcal{P} denote the projective class induced by the cotriple \mathcal{F} : $(R, A, \rho) \in \mathcal{P}$ iff there exists a morphism $\vartheta : (R, A, \rho) \rightarrow \mathcal{F}(R, A, \rho)$ such that $\tau_{(R, A, \rho)}\vartheta = 1_{(R, A, \rho)}$.

Given any crossed module (R, A, ρ) , there is an augmented simplicial object $\mathcal{F}_*(R, A, \rho) \rightarrow (R, A, \rho)$ in the category $\mathcal{X}\mathbf{Alg}$, where

$$\begin{aligned}\mathcal{F}_n(R, A, \rho) &= \mathcal{F}^{n+1}(R, A, \rho) = \mathcal{F}(\mathcal{F}^n(R, A, \rho)), \\ d_i^n &= \mathcal{F}^i(\tau_{\mathcal{F}^{n-i}}), \quad s_i^n = \mathcal{F}^i(\delta_{\mathcal{F}^{n-i}}), \quad 0 \leq i \leq n,\end{aligned}$$

and which is called the \mathcal{F} -cotriple resolution of (R, A, ρ) . Applying the functor $\mathcal{X}Ab^{add}$ dimension-wise to $\mathcal{F}_*(R, A, \rho)$ we obtain the simplicial object $\mathcal{X}Ab^{add}\mathcal{F}_*(R, A, \rho)$ in the category $\mathcal{X}\mathbf{Vect}$.

Definition 4.2.1. *The n -th cotriple cyclic homology of a crossed module of algebras (R, A, ρ) is defined by*

$$\mathcal{HC}_n(R, A, \rho) = H_n(\mathcal{X}Ab^{add}(\mathcal{F}_*(R, A, \rho))), \quad n \geq 0.$$

It is clear that \mathcal{HC}_n , $n \geq 0$ is a functor from $\mathcal{X}\mathbf{Alg}$ to $\mathcal{X}\mathbf{Vect}$. Moreover, for any $(R, A, \rho) \in \mathcal{X}\mathbf{Alg}$,

$$\mathcal{HC}_0(R, A, \rho) \cong \mathcal{X}Ab^{add}(R, A, \rho) = (R/[A, R], A/[A, A], \bar{\rho}),$$

where $\bar{\rho}$ is the linear map induced by ρ .

For further investigation of the cotriple cyclic homology of crossed modules of algebras we need some non-standard simplicial resolutions in the sense of Barr-Beck [1].

Proposition 4.2.2. *Let $((R_*, A_*, \rho_*), (d_0^0, d_0^0), (R, A, \rho))$ be an augmented simplicial crossed module of algebras. Suppose the following conditions hold:*

- (i) *the crossed module (R_n, A_n, ρ_n) , $n \geq 0$, belongs to the projective class \mathcal{P} ;*
- (ii) *the augmented simplicial algebras (R_*, d_0^0, R) and (A_*, d_0^0, A) are aspherical.*

Then the simplicial crossed modules of algebras (R_, A_*, ρ_*) and $\mathcal{F}_*(R, A, \rho)$ are homotopically equivalent.*

Proof. Straightforward from [1, 5.3]. □

Now we describe several connections between the cyclic homology of algebras and cotriple cyclic homology of crossed modules. There are two ways of regarding an algebra A as a crossed module, via the trivial map $0 : 0 \rightarrow A$ and via the identity map $1_A : A \rightarrow A$ with action of A on itself given by multiplication. Respectively there are full embeddings

$$i, \epsilon : \mathbf{Alg} \rightarrow \mathcal{X}\mathbf{Alg}$$

defined by $iA = (0, A, 0)$ and $\epsilon A = (A, A, 1_A)$. The functor i has a left adjoint $\tau : \mathcal{X}\mathbf{Alg} \rightarrow \mathbf{Alg}$, $\tau(R, A, \rho) = \text{Coker } \rho$ and also a right adjoint $\kappa : \mathcal{X}\mathbf{Alg} \rightarrow \mathbf{Alg}$, $\kappa(R, A, \rho) = A$. On the other hand, the functor κ and the functor $\xi : \mathcal{X}\mathbf{Alg} \rightarrow \mathbf{Alg}$, $\xi(R, A, \rho) = R$ are left and right adjoint to the functor ϵ , respectively.

Regarding $\mathcal{X}\mathbf{Vect}$ as a subcategory of $\mathcal{X}\mathbf{Alg}$, the cotriple cyclic homology $\mathcal{HC}_n(R, A, \rho)$, $n \geq 0$, could be presented as a linear map $\xi\mathcal{HC}_n(R, A, \rho) \rightarrow \kappa\mathcal{HC}_n(R, A, \rho)$.

Proposition 4.2.3. (i) *For any crossed module (R, A, ρ) and $n \geq 0$,*

$$\kappa\mathcal{HC}_n(R, A, \rho) \cong HC_n(A).$$

(ii) For any algebra A and $n \geq 0$,

$$\mathcal{HC}_n(iA) \cong iHC_n(A) \quad \text{and} \quad \mathcal{HC}_n(\epsilon A) \cong \epsilon HC_n(A).$$

Proof. (i) Given a crossed module (R, A, ρ) , by Proposition 4.2.2 the simplicial algebra $\kappa\mathcal{F}_*(R, A, \rho) \rightarrow A$ is a free simplicial resolution of A . Therefore

$$\kappa\mathcal{HC}_n(R, A, \rho) = \kappa H_n(\mathcal{X}Ab^{add}(\mathcal{F}_*(R, A, \rho))) = H_n(Ab^{add}(\kappa\mathcal{F}_*(R, A, \rho))).$$

The assertion follows from [7, Theorem 1.1].

(ii) Let (A_*, d_0^0, A) be a free simplicial resolution of an algebra A . It is routine to check that iA_n and ϵA_n belong to the projective class \mathcal{P} for any $n \geq 0$. Then by Proposition 4.2.2 we have

$$\mathcal{HC}_n(iA) \cong H_n(\mathcal{X}Ab^{add}(iA_*)) = H_n(iAb^{add}(A_*)) = iH_n(Ab^{add}(A_*)) \cong iHC_n(A)$$

and

$$\mathcal{HC}_n(\epsilon A) \cong H_n(\mathcal{X}Ab^{add}(\epsilon A_*)) = H_n(\epsilon Ab^{add}(A_*)) = \epsilon H_n(Ab^{add}(A_*)) \cong \epsilon HC_n(A).$$

□

Finally in this subsection we calculate the cotriple cyclic homology of an inclusion crossed module of algebras.

Proposition 4.2.4. *Let (I, A, inc) be an inclusion crossed module of algebras and $n \geq 0$. Then there is an isomorphism*

$$\xi\mathcal{HC}_n(I, A, inc) \cong HC_n(A, I),$$

where $HC_n(A, I)$ denotes the n -th relative cyclic homology.

Proof. Let $\mathcal{F}_* = \mathcal{F}_*(I, A, inc) \rightarrow (I, A, inc)$ be the cotriple resolution of (I, A, inc) . Note that both $\kappa\mathcal{F}_n$ and $\kappa\mathcal{F}_n/\xi\mathcal{F}_n$ are free algebras for each $n \geq 0$. Moreover, $\xi\mathcal{F}_* \rightarrow I$ and $\kappa\mathcal{F}_* \rightarrow A$ are aspherical augmented simplicial algebras and since (I, A, inc) is an inclusion crossed module, the augmented simplicial algebra $\kappa\mathcal{F}_*/\xi\mathcal{F}_* \rightarrow A/I$ is aspherical as well. We have the commutative diagram of complexes

$$(14) \quad \begin{array}{ccc} CC(A) & \longrightarrow & CC(A/I) \\ \uparrow & & \uparrow \\ \text{Tot}(CC(\kappa\mathcal{F}_*)) & \longrightarrow & \text{Tot}(CC(\kappa\mathcal{F}_*/\xi\mathcal{F}_*)) \\ \downarrow & & \downarrow \\ Ab^{add}(\kappa\mathcal{F}_*) & \longrightarrow & Ab^{add}(\kappa\mathcal{F}_*/\xi\mathcal{F}_*) \end{array}.$$

Clearly, by Proposition 3.1.1, the both vertical morphisms in the upper quadrant are quasi-isomorphisms. Furthermore, by [7], the both vertical morphisms in the lower quadrant are also quasi-isomorphisms. Consequently, we have an isomorphism

$$HC_n(A, I) \cong H_n(\text{Ker}\{Ab^{add}(\kappa\mathcal{F}_*) \rightarrow Ab^{add}(\kappa\mathcal{F}_*/\xi\mathcal{F}_*)\}), \quad n \geq 0.$$

Now from the five-term exact cyclic homology sequence of [17] and the fact that $HC_1(\kappa\mathcal{F}_n/\xi\mathcal{F}_n) = 0$, $n \geq 0$ [15], we deduce

$$\text{Ker}\{Ab^{add}(\kappa\mathcal{F}_*) \rightarrow Ab^{add}(\kappa\mathcal{F}_*/\xi\mathcal{F}_*)\} \cong \xi\mathcal{F}_*/[\kappa\mathcal{F}_*, \xi\mathcal{F}_*].$$

But, by definition

$$\xi\mathcal{HC}_n(I, A, inc) = \xi H_n(\mathcal{X}Ab^{add}\mathcal{F}_*) = H_n(\xi\mathcal{F}_*/[\kappa\mathcal{F}_*, \xi\mathcal{F}_*]).$$

This completes the proof. \square

Corollary 4.2.5. *Let (I, A, inc) be an inclusion crossed module of algebras. and $n \geq 1$. Then there is a long exact homology sequence*

$$(15) \quad \begin{aligned} \cdots \longrightarrow \xi\mathcal{HC}_n(I, A, inc) \longrightarrow HC_n(A) \longrightarrow HC_n(A/I) \longrightarrow \cdots \longrightarrow HC_1(A/I) \longrightarrow I/[A, I] \\ \longrightarrow A/[A, A] \longrightarrow A/(I + [A, A]) \longrightarrow 0. \end{aligned}$$

Proof. Straightforward from Proposition 4.2.4. \square

5. CYCLIC HOMOLOGY VS. COTRIPLE CYCLIC HOMOLOGY

In this section we compare two above-discussed cyclic homology theories of crossed modules of algebras in terms of long exact homology sequence. Namely, the aim of this section is to prove the following.

Theorem 5.0.6. *Let \mathbf{k} be a field of characteristic zero and (R, A, ρ) a crossed module of algebras. Then there are natural exact sequences*

$$(16) \quad \begin{aligned} \cdots \longrightarrow HC_{n+1}(R, A, \rho) \longrightarrow \xi\mathcal{HC}_n(R, A, \rho) \longrightarrow HC_n(A) \longrightarrow HC_n(R, A, \rho) \\ \longrightarrow \cdots \longrightarrow \xi\mathcal{HC}_1(R, A, \rho) \longrightarrow HC_1(A) \longrightarrow HC_1(R, A, \rho) \longrightarrow R/[A, R] \\ \xrightarrow{\rho} A/[A, A] \longrightarrow A/(\text{Im } \rho + [A, A]) \longrightarrow 0 \end{aligned}$$

and

$$(17) \quad \begin{aligned} \cdots \longrightarrow \xi\mathcal{HC}_{n-1}(R, A, \rho) \longrightarrow H_{n+1}(\beta(R, A, \rho)) \longrightarrow \xi\mathcal{HC}_n(R, A, \rho) \\ \longrightarrow \xi\mathcal{HC}_{n-2}(R, A, \rho) \longrightarrow \cdots \longrightarrow \xi\mathcal{HC}_1(R, A, \rho) \longrightarrow H_3(\beta(R, A, \rho)) \longrightarrow \xi\mathcal{HC}_2(R, A, \rho) \\ \longrightarrow \xi\mathcal{HC}_0(R, A, \rho) \longrightarrow H_2(\beta(R, A, \rho)) \longrightarrow \xi\mathcal{HC}_1(R, A, \rho) \longrightarrow 0. \end{aligned}$$

Moreover, there are isomorphisms

$$H_1(\beta(R, A, \rho)) \cong \xi\mathcal{HC}_0(R, A, \rho) \cong R/[A, R],$$

where the complex β is defined immediately below.

Note that the sequence (16) is a natural generalisation of the relative cyclic homology exact sequence (15).

5.1. The complexes beta and gamma. Given a crossed module (R, A, ρ) of algebras, we have a natural morphism of crossed modules

$$(0, 1_A) : (0, A, 0) \longrightarrow (R, A, \rho) .$$

It is easy to see that there are injective maps of bicomplexes

$$CC^{\{2\}}(\mathfrak{N}_*(0, 1_A)) : CC^{\{2\}}(\mathfrak{N}_*(0, A, 0)) \rightarrow CC^{\{2\}}(\mathfrak{N}_*(R, A, \rho))$$

and

$$CC(\mathfrak{N}_*(0, 1_A)) : CC(\mathfrak{N}_*(0, A, 0)) \rightarrow CC(\mathfrak{N}_*(R, A, \rho)),$$

which yield the respective injective maps of complexes

$$i_{(R, A, \rho)} : CC^{\{2\}}(0, A, 0) \rightarrow CC^{\{2\}}(R, A, \rho) \quad \text{and} \quad j_{(R, A, \rho)} : CC(0, A, 0) \rightarrow CC(R, A, \rho).$$

Define the complex beta, $\beta(R, A, \rho)$, and gamma, $\gamma(R, A, \rho)$, from the following commutative diagram of complexes of vectorspaces with exact rows:

$$(18) \quad \begin{array}{ccccccc} 0 & \longrightarrow & CC^{\{2\}}(0, A, 0) & \xrightarrow{i_{(R, A, \rho)}} & CC^{\{2\}}(R, A, \rho) & \longrightarrow & \beta(R, A, \rho) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & CC(0, A, 0) & \xrightarrow{j_{(R, A, \rho)}} & CC(R, A, \rho) & \longrightarrow & \gamma(R, A, \rho) \longrightarrow 0 \end{array}$$

The next two propositions calculate the low dimensional homology of β and γ complexes.

Proposition 5.1.1. *Let (R, A, ρ) be a crossed module of algebras. Then we have*

$$H_0\beta(R, A, \rho) = H_0\gamma(R, A, \rho) = 0$$

and

$$H_1\beta(R, A, \rho) = H_1\gamma(R, A, \rho) \cong R/[A, R].$$

Proof. Given a simplicial algebra A_* , the last two rows of the bicomplexes $CC^{\{2\}}(A_*)$ and $CC(A_*)$ coincide, which has the form for $\mathfrak{N}_*(0, A, 0)$

$$\begin{array}{ccccccc} A^{\otimes 2} \oplus A & \longleftarrow & A^{\otimes 2} \oplus A & \longleftarrow & A^{\otimes 2} \oplus A & \longleftarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ A & \longleftarrow & A & \longleftarrow & A & \longleftarrow & , \end{array}$$

while for $\mathfrak{N}_*(R, A, \rho)$

$$\begin{array}{ccccccc} A^{\otimes 2} \oplus A & \longleftarrow & (R \rtimes A)^{\otimes 2} \oplus (R \rtimes A) & \longleftarrow & (R \rtimes R \rtimes A)^{\otimes 2} \oplus (R \rtimes R \rtimes A) & \longleftarrow & \\ \downarrow & & \downarrow & & \downarrow & & \\ A & \longleftarrow & R \rtimes A & \longleftarrow & R \rtimes R \rtimes A & \longleftarrow & . \end{array}$$

Hence, it is clear that we have

$$H_0\beta(R, A, \rho) = H_0\gamma(R, A, \rho) = 0.$$

Moreover, comparing the given rows of the bicomplexes, we simply deduce that

$$H_1\beta(R, A, \rho) = H_1\gamma(R, A, \rho) = \text{Coker} \left((R \rtimes A)^{\otimes 2} / A^{\otimes 2} \longrightarrow (R \rtimes A) / A \right),$$

where the arrow is defined as follows:

$$(r, a) \otimes (r', a') \mapsto (r, a)(r', a') - (r', a')(r, a) = ([r, r'] + [r, a'] + [a, r'], [a, a']).$$

This implies the second isomorphism of the assertion. \square

Given an algebra A , by Corollary 3.1.2 the following pairs of complexes $CC^{\{2\}}(0, A, 0)$, $CC^{\{2\}}(A)$ and $CC(0, A, 0)$, $CC(A)$ are quasi-isomorphic. Then taking into account Proposition 5.1.1, for any crossed module (R, A, ρ) of algebras, the diagram (18) induces the morphism of long exact homology sequences

$$(19) \quad \begin{array}{ccccccc} \longrightarrow & HH_n(A) & \longrightarrow & HH_n(R, A, \rho) & \longrightarrow & H_n\beta(R, A, \rho) & \longrightarrow \cdots \longrightarrow H_2\beta(R, A, \rho) \\ & \downarrow & & \downarrow & & \downarrow & \downarrow \\ \longrightarrow & HC_n(A) & \longrightarrow & HC_n(R, A, \rho) & \longrightarrow & H_n\gamma(R, A, \rho) & \longrightarrow \cdots \longrightarrow H_2\gamma(R, A, \rho) \end{array}$$

$$\begin{array}{ccccccc} \longrightarrow & HH_1(A) & \longrightarrow & HH_1(R, A, \rho) & \longrightarrow & R/[A, R] & \longrightarrow HH_0(A) \longrightarrow HH_0(R, A, \rho) \longrightarrow 0 \\ & \downarrow & & \downarrow & & \parallel & \parallel \\ \longrightarrow & HC_1(A) & \longrightarrow & HC_1(R, A, \rho) & \longrightarrow & R/[A, R] & \longrightarrow HC_0(A) \longrightarrow HC_0(R, A, \rho) \longrightarrow 0. \end{array}$$

Proposition 5.1.2. *Let $(R, A, \rho) \in XAlg$ belong to the projective class \mathcal{P} (see Subsection 4.2). Then*

- (i) $H_n\beta(R, A, \rho) = 0$ for any $n > 2$;
- (ii) if in addition \mathbf{k} is a field of characteristic zero, we have

$$H_n\gamma(R, A, \rho) = 0 \quad \text{for any } n > 1$$

and an isomorphism

$$H_2\beta(R, A, \rho) \cong R/[A, R].$$

Proof. Without loss of generality we can assume that $(R, A, \rho) = \mathbb{F}(V)$ for some $V \in \mathbf{Vect}$. Hence $R \xrightarrow{\rho} A$ is an inclusion crossed module with A and A/R being free algebras. Then, using again Corollary 3.1.2, the pairs of complexes $CC^{\{2\}}(R, A, \rho)$, $CC^{\{2\}}(A/R)$ and $CC(R, A, \rho)$, $CC(A/R)$ are quasi-isomorphic.

Now (i) follows directly from the top exact sequence of the diagram (19) and the fact that, for a free algebra F , the Hochschild homology $HH_n(F)$ vanishes for any $n \geq 2$.

If in addition \mathbf{k} is a field of characteristic zero, then the cyclic homology $HC_n(F)$ of a free algebra F vanishes as well for any $n \geq 1$ (see [15, Proposition 5.4]). Therefore the

diagram (19) implies that $H_n \gamma(R, A, \rho) = 0$ for $n > 1$. Besides, by the Connes' periodic exact sequence there is a natural isomorphism

$$HC_0(F) \xrightarrow{\cong} HH_1(F).$$

Then the five term exact sequence in cyclic homology from [17] implies the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R/[A, R] & \longrightarrow & HC_0(A) & \longrightarrow & HC_0(R, A, \rho) \cong HC_0(A/R) & \longrightarrow & 0 \\ & & \downarrow & & \cong \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & H_2 \beta(R, A, \rho) & \longrightarrow & HH_1(A) & \longrightarrow & HH_1(R, A, \rho) \cong HH_1(A/R) & \longrightarrow & 0 \end{array}$$

which completes the proof. \square

5.2. Proof of Theorem 5.0.6. Given a simplicial crossed module (R_*, A_*, ρ_*) and a functor $\Phi : \mathcal{X}\mathbf{Alg} \rightarrow \mathfrak{C}_{\geq 0}$, denote by $\Phi(R_*, A_*, \rho_*)$ the bicomplex of vectorspaces obtained by applying the functor Φ dimension-wise to the simplicial crossed module (R_*, A_*, ρ_*) .

The following lemma will be needed.

Lemma 5.2.1. *Let $((R_*, A_*, \rho_*), (d_0^0, d_0^0), (R, A, \rho))$ be an augmented simplicial crossed module of algebras. Suppose (R_*, d_0^0, R) and (A_*, d_0^0, A) are aspherical augmented simplicial algebras. Then the augmented simplicial vectorspaces*

$$(\beta_n(R_*, A_*, \rho_*), \beta_n(d_0^0, d_0^0), \beta_n(R, A, \rho)) \quad \text{and} \quad (\gamma_n(R_*, A_*, \rho_*), \gamma_n(d_0^0, d_0^0), \gamma_n(R, A, \rho))$$

are acyclic for any $n \geq 0$.

Proof. We shall prove only the acyclicity of the second augmented vectorspace. The proof for the first is similar.

In fact, using the fact that the semi-direct product of aspherical simplicial algebra is aspherical as well, the augmented simplicial algebra

$$(E_q(R_*, A_*, \rho_*), E_q(d_0^0, d_0^0), E_q(R, A, \rho)), \quad q \geq 0,$$

is aspherical. Now by Proposition 3.1.1 (i) we have that the augmented simplicial vectorspace

$$(CC_p(E_q(R_*, A_*, \rho_*)), CC_p(E_q(d_0^0, d_0^0)), CC_p(E_q(R, A, \rho))), \quad p \geq 0, q \geq 0$$

is acyclic. This clearly implies that the augmented simplicial vectorspaces

$$(CC_n(R_*, A_*, \rho_*), CC_n(d_0^0, d_0^0), CC_n(R, A, \rho))$$

and consequently

$$(CC_n(0, A_*, 0), CC_n(0, d_0^0), CC_n(0, A, 0))$$

are acyclic for any $n \geq 0$. Clearly, the short exact sequence of augmented simplicial vectorspaces

$$\begin{array}{ccccccc}
0 & \longrightarrow & CC_n(0, A_*, 0) & \longrightarrow & CC_n(R_*, A_*, \rho_*) & \longrightarrow & \gamma_n(R_*, A_*, \rho_*) \longrightarrow 0 \\
& & \downarrow CC_n(0, d_0^0) & & \downarrow CC_n(d_0^0, d_0^0) & & \downarrow \gamma_n(d_0^0, d_0^0) \\
0 & \longrightarrow & CC_n(0, A, 0) & \longrightarrow & CC_n(R, A, \rho) & \longrightarrow & \gamma_n(R, A, \rho) \longrightarrow 0,
\end{array}$$

implies the result. \square

Return to the proof of Theorem 5.0.6.

Consider the bicomplex $\gamma(R_*, A_*, \rho_*)$, where $\left((R_*, A_*, \rho_*), (d_0^0, d_0^0), (R, A, \rho)\right)$ is a simplicial resolution of (R, A, ρ) in $\mathcal{X}\mathbf{Alg}$ in the sense of Barr-Beck [1] (see Proposition 4.2.2). For any fixed q the homology of the complex $\gamma_q(R_*, A_*, \rho_*)$, by Lemma 5.2.1, is $H_p(\gamma_q(R_*, A_*, \rho_*)) = 0$ if $p > 0$ and $H_0(\gamma_q(R_*, A_*, \rho_*)) \cong \gamma_q(R, A, \rho)$. Therefore $H_n(\text{Tot}(\gamma(R_*, A_*, \rho_*))) \cong H_n(\gamma(R, A, \rho))$. On the other hand, there is a spectral sequence

$$E_{pq}^1 = H_q(\gamma(R_p, A_p, \rho_p)) \implies H_{p+q}(\gamma(R, A, \rho)).$$

But by Proposition 5.1.1 and Proposition 5.1.2 we have

$$E_{pq}^1 = \begin{cases} 0 & \text{for } p \geq 0, q \neq 1 \\ R_p/[A_p, R_p] & \text{for } p \geq 0, q = 1. \end{cases}$$

Moreover, $E_{pq}^2 \cong \xi\mathcal{HC}_p(R, A, \rho)$ for $q = 1, p \geq 0$. Then the degenerated spectral sequence E_{pq}^2 yields the natural isomorphism

$$(20) \quad H_{n+1}(\gamma(M, R, \mu)) \cong \xi\mathcal{HC}_n(M, R, \mu), \quad n \geq 0.$$

Thus (19) and (20) imply the exact sequence (16). Furthermore, from (18) one easily deduces that for any crossed module (R, A, ρ) there is a short exact sequence of complexes

$$0 \longrightarrow \beta(R, A, \rho) \longrightarrow \gamma(R, A, \rho) \longrightarrow \gamma(R, A, \rho)[2] \longrightarrow 0,$$

where $\gamma(R, A, \rho)[2]$ is the dimension shifted complex by 2, i.e. $\gamma_n(R, A, \rho)[2] = \gamma_{n-2}(R, A, \rho)$, $n \geq 0$. Now the induced long exact homology sequence with the isomorphism (20) completes the rest part of the theorem.

REFERENCES

- [1] M. Barr and J. Beck, *Homology and Standard Constructions*, Lecture Notes in Mathematics, Vol. **80**, Springer-Verlag, Berlin/New York (1969), 245–335.
- [2] H.-J. Baues, “*Combinatorial homotopy and 4-dimensional complexes*”, de Gruyter, Berlin, 1991.
- [3] H.-J. Baues and E. G. Minian, *Crossed extensions of algebras and Hochschild cohomology*, Homology Homotopy Appl. **4** (2) (2002), 63–82.
- [4] R. Brown, *Groupoids and crossed objects in algebraic topology*, Homology Homotopy Appl. **1** (1) (1999), 1–78.
- [5] P. Carrasco, A.M. Cegarra and A.R. Grandjeán, *(Co)homology of crossed modules*, J. Pure Appl. Algebra **168** (2002), 147–176.
- [6] P. Dedecker and A.S.-T. Lue, *A nonabelian two-dimensional cohomology for associative algebras*, Bull. Amer. Math. Soc. **72** (6) (1966), 1044–1050.

- [7] G. Donadze, N. Inassaridze and M. Ladra, *Derived functors and Hopf type formulas in cyclic homology*, Bulletin de SMF (2008) (to appear).
- [8] G. J. Ellis, *Higher dimensional crossed modules of algebras*, J. Pure Appl. Algebra **52** (1988), 277–282.
- [9] G. J. Ellis, *Homology of 2-types*, J. London Math. Soc. (2) **46** (1992), 1–27.
- [10] A.R. Grandjeán, M. Ladra and T. Pirashvili, *CCG-homology of crossed modules via classifying spaces*, J. Algebra **229** (2000), 660–665.
- [11] S. Geller and C. Weibel, *Hochschild and cyclic homology are far from being homotopy functors*, Proc. Amer. Math. Soc. **106** (1) (1989), 49–57.
- [12] H. Inassaridze, *Non-abelian homological algebra and its applications*, Kluwer Academic Publishers, Dordrecht, 1997.
- [13] N. Inassaridze and M. Ladra, *Hopf type formulas for cyclic homology*, C. R. Math. Acad. Sci. Paris, Ser.I **346** (2008), 385–390.
- [14] C. Kassel and J.-L. Loday, *Extensions centrales d’algèbres de Lie*, Ann. Inst. Fourier (Grenoble) **32** (1982), 119–142.
- [15] J.-L. Loday and D. G. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984) 565–591.
- [16] A.S.-T. Lue, *Non abelian cohomology of associative algebras*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 159–180.
- [17] D. Quillen, *Cyclic cohomology and algebra extensions*, K-Theory **3** (1989), 205–246.
- [18] J. H. C. Whitehead, *Combinatorial homotopy II*, Bull. Amer. Math. Soc. **55** (1949), 453–496.
- [19] M. Wodzicki, *Excision in cyclic homology and in rational algebraic K-theory*, Ann. Math. (2) **129** (1989), 591–639.

Guram Donadze: Department of Algebra, A.Razmadze Mathematical Institute, M.Alexidze St. 1, 0193 Tbilisi, Georgia

E-mail address: donad@rmi.acnet.ge

Nick Inassaridze: Department of Algebra, A.Razmadze Mathematical Institute, M.Alexidze St. 1, 0193 Tbilisi, Georgia

Current address: Departamento de Matemática Aplicada I, Universidad de Vigo, EUIT Forestal, 36005 Pontevedra, Spain

E-mail address: niko.inas@gmail.com

Emzar Khmaladze: Department of Algebra, A.Razmadze Mathematical Institute, M.Alexidze St. 1, 0193 Tbilisi, Georgia

E-mail address: e.khmal@gmail.com

Manuel Ladra: Departamento de Álgebra, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago de Compostela, Spain

E-mail address: manuel.ladra@usc.es